Asymptotic estimation of some multiple integrals and the electromagnetic deuteron form factors at high momentum transfer

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Abstract

A theorem about asymptotic estimation of multiple integral of a special type is proved for the case when the integrand peaks at the integration domain bound, but not at a point of extremum. Using this theorem the asymptotic expansion of the electromagnetic deuteron form factors at high momentum transfers is obtained in the framework of two-nucleon model in both nonrelativistic and relativistic impulse approximations. It is found that relativistic effects slow down the decrease of deuteron form factors and result in agreement between the relativistic asymptotics and experimental data at high momentum transfers.

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1 Introduction

Recent advances in experimental investigations of hadron structure arouse the interest in the theoretical study of the hadron electromagnetic form factors at high momentum transfers (see e.g. review [1] and references therein). In this connection the JLab program of investigations on elastic electron-deuteron scattering experiments at $Q^2 \simeq 10$ (GeV/c)^2 ($Q^2 = -q^2$, q is transferred momentum) [2] attracts considerable attention. There exists a hope that these JLab experiments will help to determine the limits of application for the two-nucleon model and to clarify the interplay between nucleon-nucleon and quark approaches to the deuteron.

Using the asymptotic expansion presented in this paper we show [3] that the momentum transfer region in the JLab experiments is asymptotical for the deuteron considered as a nucleon–nucleon system. That is why the study of the electromagnetic deuteron form factors is interesting at $Q^2 \to \infty$.

The present work is devoted to the theoretical investigation of deuteron form factors asymptotic behavior at high momentum transfer in the framework of the two-nucleon model. The form factors asymptotics is studied in accordance with the next points.

1. As a rule, calculation of the form factors asymptotic behavior in the relativistic approaches reduces to the asymptotic estimation of n-tuple integrals. In the relativistic approach used in our work the deuteron form factors are expressed in terms of double integrals where integrands peak at the integration domain bound, and corresponding point is not a point of extremum. In this connection the theorem defining asymptotic expansion of n-tuple integrals with such integrand is proven in our paper.

2. In general, high momentum transfers require relativistic consideration. However we begin the consideration of the asymptotic estimation of electromagnetic deuteron form factors with the nonrelativistic case and nonrelativistic impulse approximation at $Q^2 \to \infty$. This is due to the facts that, at first, the nonrelativistic calculation is a less complicated and, second, this calculation is important for establishment of the role of the relativistic effects.

3. The asymptotic expansion of the relativistic deuteron form factors is calculated in the relativistic invariant impulse approximation in a variant of instant form of Poincare-invariant quantum mechanics (PIQM) developed in our papers previously [4, 5, 6, 7, 8]. The relativistic calculations are performed by analogy with nonrelativistic case. It is shown that relativistic effects essentially slow down the asymptotical decrease of the form factors.

4. It is found that obtained in the framework of the two-nucleon model relativistic asymptotics coincides with the experimental data.

This paper is organized as follows. Section 2 is devoted to the proving of the central for this work theorem defining asymptotics of multiple integrals of some special type. In Sec. 3 a brief review of the formulas for the deuteron form factors in the nonrelativistic and relativistic invariant impulse approximation is given. The deuteron form factors asymptotics is calculated in nonrelativistic and relativistic impulse approximation with the help of the proven theorem in Sec. 4. In Sec. 5 asymptotics of the form factors is obtained for the deuteron wave functions in the conventional representation as a discrete superposition of Yukawa-type terms [9]. Sec. 6 contains the conclusions of this paper.

2 Theorem on the asymptotic expansion of some multiple integrals in the case when the maximal value of the integrand belongs to region boundary

In the following we will consider integrals of the kind:

$$F(\lambda) = \int_{\Omega} f(\lambda, x) \exp[S(\lambda, x)] dx \;, \tag{1}$$

where Ω is a domain in $\mathbf{R}^n$, $x = (x_1, ..., x_n)$, λ is a large positive parameter. We will use following definitions: $\partial\Omega$ is a bound of the domain Ω, $[\Omega] = \Omega \cup \partial\Omega$, the bound $\partial\Omega \in C^\infty$ if in the vicinity of any point $x^0 \in \partial\Omega$ it can be specified by equation $x_j = \varphi(x'),\ x' \in U',\ x' = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$, U' is a neighborhood of a point ${x'}^0$, and the function $\varphi(x') \in C^\infty$ in U'.

Let us prove now a lemma and a theorem on the asymptotic estimation of integrals in Eq.(1).

L e m m a. *Let $S(\lambda, x)$ be a smooth function in the $[\Omega]$, $f(\lambda, x)$ be a continuous function in the $[\Omega]$, and $M(\lambda) \in C^1$,*

$$M(\lambda) = \sup_{x \in [\Omega]} S(\lambda, x) < \infty \;,$$

at some $\lambda_0 > 0$ the integral (1) be absolutely convergent:

$$\int_{\Omega} |f(\lambda_0, x)| \exp[S(\lambda_0, x)] dx < \infty \;,$$

and the following conditions be fulfilled at $\lambda \geq \lambda_0$:

$$\frac{\partial S(\lambda, x)}{\partial \lambda} \leq \frac{dM(\lambda)}{d\lambda} \;, \tag{2}$$

$$|f(\lambda, x)| \leq C_1 |f(\lambda_0, x)| \;. \tag{3}$$

Then at $\lambda \geq \lambda_0$ the following estimation is valid:

$$|F(\lambda)| \leq C_2 e^{M(\lambda)} \;. \tag{4}$$

P r o o f. At $\lambda \geq \lambda_0$ the following estimations are true:

$$|F(\lambda)| \leq e^{M(\lambda)} \int_{\Omega} e^{S(\lambda_0,x)-M(\lambda_0)} e^{S(\lambda,x)-S(\lambda_0,x)-M(\lambda)+M(\lambda_0)} |f(\lambda, x)| dx \leq$$

$$\leq e^{M(\lambda)-M(\lambda_0)} \int_{\Omega} e^{S(\lambda_0,x)+S(\lambda,x)-S(\lambda_0,x)-M(\lambda)+M(\lambda_0)} |f(\lambda, x)| dx.$$

From conditions (2),(3) we obtain the inequality:

$$|F(\lambda)| \leq C_1 e^{M(\lambda)-M(\lambda_0)} \int_{\Omega} e^{S(\lambda_0,x)} |f(\lambda_0, x)| dx \leq C_2 e^{M(\lambda)} \;.$$

Thus the statement (4) of the lemma is proved.

Later we will consider function $S(\lambda, x)$ described in lemma which has the maximal value in the point $x^0 \in \partial\Omega$, and $S(\lambda, x)$, $\partial\Omega \in C^\infty$ in the vicinity of x^0. This point is not the point of extremum, that means the validity of the following conditions:

$$\frac{\partial S(\lambda, x^0)}{\partial n} \neq 0 \, , \tag{5}$$

and matrix of coefficients B:

$$\left\| \frac{\partial^2 S(\lambda, x^0)}{\partial \xi_i \partial \xi_j} \right\|_{i,j=1}^{n-1} = B \, , \tag{6}$$

gives the negative determined quadratic form. In Eqs. (5), (6) $\partial/\partial n$ specifies the internal normal derivative $\vec{n}$ to the $\partial\Omega$, and $\xi_1, ..., \xi_{n-1}$ is an orthonormal basis in the tangential to the $\partial\Omega$ plane $T\,\partial\Omega_{x^0}$ at the x^0 point.

For convenience let us choose in the vicinity of point x^0 a frame $y = (y_1, ..., y_n)$, so that x^0 is the origin of coordinate and the internal normal to $\partial\Omega$ coincides with the last basis vector of the new coordinate system. Functions f, S in these coordinates we denote as f^*, S^*, and U^* is an image of U (that is an image of a half-vicinity of the point x^0). The equation for ∂U^* in the vicinity of the point $y = 0$ can be written in the following way:

$$y_n = \varphi(y'), \qquad y' \in U', \qquad y' = (y_1, ..., y_{n-1}),$$

with U' is a vicinity of the point $y' = 0$, $\varphi(y') \in C^\infty(U')$, and at $y' \to 0, \varphi(y') = O(|\, y' \,|^2)$.

T h e o r e m. *Let the following conditions be fulfilled:*

1°. $f, S \in C([\Omega])$.

2°. $S(\lambda, x^0)$ *is maximal value of function* $S(\lambda, x)$, $x^0 \in \partial\Omega$, *and* x^0 *is not point of extremum.*

3°. $f,\ S,\ \partial\Omega \in C^\infty$ *in the vicinity of the point* x^0.

4°. *The Taylor expansion of functions* S^* *and* f^* *in the vicinity of point* x^0 *satisfy the following relations:*

$$f^*(\lambda, y) = f^*(\lambda, 0)[1 + o(1)] \, , \tag{7}$$

$$S^*(\lambda, y', \varphi(y')) - S^*(\lambda, 0) = \frac{1}{2}\langle Ay',\ y'\rangle + O(|\, y' \,|^3) \, , \tag{8}$$

the matrix $A = \left\| \frac{\partial^2 S^*(\lambda,0)}{\partial y_i \partial y_j} \right\|_{i,j=1}^{n-1}$, *angle brackets denote bilinear form:* $\langle x,\ y\rangle = x_1 y_1 + x_2 y_2 + ... + x_n y_n$.

Then at $\lambda \to \infty$ *the following asymptotic expansion is valid:*

$$F(\lambda) \sim \exp[S(\lambda, x^0)] \sum_{k=0}^{\infty} a_k(\lambda) \, . \tag{9}$$

The way to calculate coefficients $a_k(\lambda)$ *will be determined later.*

P r o o f. Let us divide the integral (1) into two integrals. Integration domain of the first one is the half-vicinity U of the point x^0, and integration domain of the second one is a remainder of integration domain of the original integral. It is easily shown by the proven lemma that the second integral is exponentially small as compared with $\exp[S(\lambda, x^0)]$. So we will estimate asymptotically the first integral only.

In the expansion of the function $S^*(\lambda, y', \varphi(y'))$ in line with the condition (8) linear components are absent, because the point $y' = 0$ is a point of maximum of the function $S^*(\lambda, y', \varphi(y'))$ in the region U'.

Let us choose U in accordance with inequalities $\varphi(y') \le y_n \le \delta,\ \delta > 0$ at $y \in U^*$. Then we can represent the integral (1) within exponentially decreasing terms:

$$F(\lambda) = \int\limits_{U^*} f^*(\lambda, y) \exp[S^*(\lambda, y)] dy \ . \tag{10}$$

Let us rewrite integral in Eq. (10) in the following way:

$$F(\lambda) = \int\limits_{U'} \Phi(\lambda, y') dy' \ ,$$

with

$$\Phi(\lambda, y') = \int\limits_{\varphi(y')}^{\delta} \exp[S^*(\lambda, y)] f^*(y) dy_n \ . \tag{11}$$

The integral (11) is one-dimensional, and the function $S^*(\lambda, y)$ reaches extremum on the boundary $y_n = \varphi(y')$. Asymptotic expansion of this integral can be found through integration by parts. After $N+1$ integration we obtain the sequence:

$$\Phi(\lambda, y') = \sum_{k=0}^{N} M^k \left[\frac{f^*(\lambda, y)}{S^{*\prime}(\lambda, y)} \right] \exp[S^*(\lambda, y)]|_{\varphi(y')}^{\delta} - \int\limits_{\varphi(y')}^{\delta} M^N \left[\frac{f^*(\lambda, y)}{S^{*\prime}(\lambda, y)} \right]' \exp[S^*(\lambda, y)] dy_n \ ,$$

with M^0 is a unit operator and

$$M^k = - \frac{1}{S^{*\prime}(\lambda, y)} \frac{d^k}{dy_n^k} \ .$$

The substitution of $y_n = \varphi(y')$ provides the main contribution to the asymptotics, the value of $y_n = \delta$ is exponentially small as compared with the previous. Further integration under these conditions gives the following expansion for the function (11):

$$\Phi(\lambda, y') = - \exp[S^*(\lambda, y', \varphi(y'))] \sum_{k=0}^{\infty} M^k \left[\frac{f^*(\lambda, y', \varphi(y'))}{S^{*\prime}(\lambda, y', \varphi(y'))} \right] \ .$$

So

$$F(\lambda) = - \sum_{k=0}^{\infty} \int\limits_{U'} \exp[S^*(\lambda, y', \varphi(y'))] M^k \left[\frac{f^*(\lambda, y', \varphi(y'))}{S^{*\prime}(\lambda, y', \varphi(y'))} \right] dy' \tag{12}$$

The point $y' = 0$ is an internal point of maximum of the integrand in the expression (12). Functions $S^*(\lambda, y', \varphi(y'))$ and $f^*(\lambda, y', \varphi(y'))$ satisfy the conditions of lemma (2) and theorem (7), (8), therefore we can apply a formula for asymptotic estimation of the n-tuple Laplas integrals [10] and obtain corresponding asymptotic expansion (9) of the integral in Eq. (12). Thus the theorem is proven.

Note, that in the general case it is rather difficult to write a compact formula for the coefficients $a_k(\lambda)$ in Eq. (9). They can be obtained such kind of way in any specific cases.

As an example, these coefficients will be obtained and written out explicitly for double integrals in the consideration of the asymptotic estimation of electromagnetic deuteron form factors in Sec. 4. Here we write out only the first asymptotic term from Eq. (9) in the x variables:

$$F(\lambda) \;\sim\; -(2\pi)^{\frac{n-1}{2}} \exp[S(\lambda, x^0)] \left(\frac{\partial S(\lambda, x^0)}{\partial n}\right)^{-1} |\det B|^{-\frac{1}{2}} f(\lambda, x^0) \;, \tag{13}$$

where $\vec{n}$, B are defined by conditions (5) and (6).

3 Electromagnetic deuteron form factors in the nonrelativistic and relativistic impulse approximation

In the nonrelativistic impulse approximation known formulas for electromagnetic deuteron form factors can be rewritten in the following way [11]:

$$G_C^{NR}(Q^2) = \sum_{l,l'} \int k^2\,dk\,k'^{\,2}\,dk'\,u_l(k)\,\tilde{g}_{0C}^{ll'}(k\,,Q^2\,,k')\,u_{l'}(k')\;,$$

$$G_Q^{NR}(Q^2) = \frac{2\,M_d^2}{Q^2} \sum_{l,l'} \int k^2\,dk\,k'^{\,2}\,dk'\,u_l(k)\,\tilde{g}_{0Q}^{ll'}(k\,,Q^2\,,k')\,u_{l'}(k')\;,$$

$$G_M^{NR}(Q^2) = -\,M_d \sum_{l,l'} \int k^2\,dk\,k'^{\,2}\,dk'\,u_l(k)\,\tilde{g}_{0M}^{ll'}(k\,,Q^2\,,k')\,u_{l'}(k')\;. \tag{14}$$

Here $u_l(k)$ are the deuteron wave functions in momentum representation, l , $l' = 0{,}2$ are orbital angular momenta, $\tilde{g}_{0i}^{ll'}(k\,,Q^2\,,k')$, $i = C, Q, M$ are nonrelativistic free two-particles charge, quadrupole and magnetic dipole form factors, M_d is the deuteron mass. Formulas for $\tilde{g}_{0i}^{ll'}$ are given in [8].

Let us discuss briefly possible types of the model deuteron wave functions. There are several classes of the deuteron wave functions: obtained with microscopic model Hamiltonians of the NN-interaction in the non-relativistic nuclear physics (for example, see [9]), deduced from scattering amplitudes in the Bethe-Salpeter approach and its various quasipotential reductions (see [12]), wave functions of the Poincare-invariant quantum mechanics (as an example see wave functions in the instant form of PIQM [4, 5, 6, 7, 8]), and also wave functions calculated in the various statements of inverse scattering problems [13, 14, 15]. But independently of the method any wave function can be represented as the following Laguerre polynomial expansion [15]:

$$u_l(k) = \sum_{m=0}^{\infty} a_{lm} \sqrt{\frac{2m!}{\Gamma(m+l+3/2)}}\; r_0^{l+\frac{3}{2}}\; k^l\; L_m^{l+\frac{1}{2}}(r_0^2 k^2)\; e^{-\frac{r_0^2 k^2}{2}} \tag{15}$$

or in the coordinate representation:

$$u_l(r) = \sum_{m=0}^{\infty} (-1)^m a_{lm} \sqrt{\frac{2m!}{r_0\,\Gamma(m+l+3/2)}}\; \left(\frac{r}{r_0}\right)^{l+1}\; L_m^{l+\frac{1}{2}}\left(\frac{r^2}{r_0^2}\right)\; e^{-\frac{r^2}{2r_0^2}}\;, \tag{16}$$

here $L_m^{l+1/2}(x)$ are generalized Laguerre polynomials, $\Gamma(x)$ is an Euler gamma function, the dimensional parameter r_0 can be related to the deuteron matter radius (see Sec. 5).

The wave function representation as a Laguerre polynomial expansion (15) is very useful for the calculation of the asymptotic behavior of the form factors. However, one can avoid such representation and obtain the asymptotic expansion directly for the initial wave function.

Generally, at high transferred momentum it is necessary to take into account relativistic corrections in the electromagnetic deuteron structure. In our paper relativistic description of the deuteron is constructed in the framework of instant form of Poincare-invariant quantum mechanics (PIQM), developed by authors previously [4, 5, 6, 7, 8]. In this approach we present electromagnetic deuteron form factors by analogy with nonrelativistic case (14). Corresponding formulas in the relativistic impulse approximation were obtained in our paper [6]:

$$G_C^R(Q^2) = \sum_{l,l'} \int d\sqrt{s}\, d\sqrt{s'}\, \varphi_l(s)\, g_{0C}^{ll'}(s\,, Q^2\,, s')\, \varphi_{l'}(s')\;,$$

$$G_Q^R(Q^2) = \frac{2\,M_d^2}{Q^2} \sum_{l,l'} \int d\sqrt{s}\, d\sqrt{s'}\, \varphi_l(s)\, g_{0Q}^{ll'}(s\,, Q^2\,, s')\, \varphi_{l'}(s')\;,$$

$$G_M^R(Q^2) = -\,M_d \sum_{l,l'} \int d\sqrt{s}\, d\sqrt{s'}\, \varphi_l(s)\, g_{0M}^{ll'}(s\,, Q^2\,, s')\, \varphi_{l'}(s')\;, \qquad (17)$$

where $\varphi_l(s)$ are the deuteron wave functions in sense of PIQM, $g_{0i}^{ll'}((s\,, Q^2\,, s')$, $i = C, Q, M$ are relativistic free two-particles charge, quadrupole and magnetic dipole form factors. Formulas for free form factors are given in [8].

The deuteron wave functions in sense of PIQM are solutions of eigenvalue problem for a mass squared operator for the deuteron (see, e.g. [4]): $\hat{M}_d^2\,|\psi\rangle = M_d^2\,|\psi\rangle$. An eigenvalue problem for this operator is coincident with the nonrelativistic Schrödinger equation within a second order on deuteron binding energy $\varepsilon_d^2/(4M)$, the value of which is small (M is an averaged nucleon mass). So the deuteron wave functions in sense of PIQM differ from nonrelativistic wave functions by conditions of normalization only. In the relativistic case the wave functions are normalized with relativistic density of states:

$$\sum_{l=0,2} \int_0^\infty \varphi_l^2(k)\, \frac{dk}{2\sqrt{k^2 + M^2}} = 1\;, \quad \varphi_l(k) = \sqrt[4]{s}\, k\, u_l(k)\;, \quad s = 4(k^2 + M^2)\;. \qquad (18)$$

Nonrelativistic formulas Eq. (14) can be obtained from relativistic ones (17) in the nonrelativistic limit. This limiting procedure can be performed in the most natural way in the instant form of PIQM. The reason is that in papers [4, 5, 6, 7, 8] we have constructed the successful formalism of the instant form of PIQM. In the case of other forms of PIQM (point and front forms) the obtaining of nonrelativistic limit is much more difficult.

For obtaining the asymptotic form factors behavior at high transferred momentum in the nonrelativistic and relativistic cases it is necessary to estimate asymptotically double integrals (14) and (17) at $Q^2 \to \infty$. Notice that integrands reach its maximum value at the integration domain bound, and this point is not a point of extremum. In the previous Section the theorem defining asymptotics of n-tuple integrals of such kind was proven.

4 Asymptotic expansion of the deuteron form factors

We start asymptotic expansion of the deuteron form factors from the nonrelativistic case. It is caused by the simplicity of the nonrelativistic formulas, so the calculation of the asymptotics is more clear. In what follows the relativistic calculation will be presented analogous to the nonrelativistic one, although more combersome. Moreover nonrelativistic calculation is interesting because nonrelativistic formulas for the form factors (14) are conventional, that is why its correct asymptotic expansion has universal significance. Let us emphasize also that the relativistic expressions for form factors and, therefore, their asymptotic estimations depend on the choice of the method of relativisation of the two-nucleon model. Nonrelativistic calculation is also of interest because it helps to clarify the role of relativistic effects in the electromagnetic structure of the deuteron at the asymptotical momentum transfers.

As we have seen in Sec. 3, the deuteron form factors in the nonrelativistic impulse approximation can be represented by double integrals (14). We will find its asymptotic expansion using the theorem of Section 2 and use as an example the asymptotics of the charge form factor. We shall estimate only the $l = l' = 0$ term in the sum (14) because the asymptotics of the other terms of form factors (14) can be derived analogously.

Let us rewrite the corresponding $l = l' = 0$ term of the charge form factor (14) using Eq. (15):

$$\int \tilde g^{00}_{0C}(k\,,Q^2\,,k')\,\exp\left[S(k,k')\right]\,k^2\,dk\,k'^{\,2}\,dk'\,\times$$

$$\times\left(\sum_m a_{0m}\sqrt{\frac{2m!}{\Gamma(m+3/2)}}\;r_0^{\frac{3}{2}}\;L_m^{\frac{1}{2}}(r_0^2k^2)\right)\left(\sum_m a_{0m}\sqrt{\frac{2m!}{\Gamma(m+3/2)}}\;r_0^{\frac{3}{2}}\;L_m^{\frac{1}{2}}(r_0^2k'^{\,2})\right)\;. \quad (19)$$

We have denoted in (19):

$$S(k,k') = -\frac{r_0^2}{2}\,(k^2+k'^2)\;. \quad (20)$$

The expression for $\tilde g^{00}_{0C}(k\,,Q^2\,,k')$ is commonly accepted (see, e.g., [8]):

$$\tilde g^{00}_{0C}(k,Q^2,k') = \frac{1}{k\,k'\,Q}\left[\theta\left(k'-\left|k-\frac{Q}{2}\right|\right)-\theta\left(k'-k-\frac{Q}{2}\right)\right]\left(G^p_E(Q^2)+G^n_E(Q^2)\right), \quad (21)$$

$G^{p,n}_E(Q^2)$ are electric form factors of proton and neuteron respectively, $\theta(x)$ is a step function.

In the case under consideration the space dimension $n = 2$, $(x_1, x_2) = (k, k')$, $\lambda = Q^2$ is a large positive parameter. Integration domain is determined by θ-functions in Eq. (21) and shown in Fig.1. The location of the point of maximal value of the function S can be obtained by analysis of (20) and (21): $(k^0, k'^{\,0}) = (\frac{Q}{4}, \frac{Q}{4})$.

Let us perform the transition to the new basis as we have descripted before. We perform the shift of the origin of coordinates to the point of maximal value of the function S. Then we rotate the obtained frame for the internal normal to the boundary in the new origin to be coincident with the last basis vector of the new frame. This procedure is illustrated in Fig.1.

In the other words we perform the transition to the new variables in Eq. (19):

$$k = \frac{1}{\sqrt{2}}\,(t'+t)+\frac{Q}{4}\;,\quad k' = \frac{1}{\sqrt{2}}\,(t'-t)+\frac{Q}{4}\;. \quad (22)$$

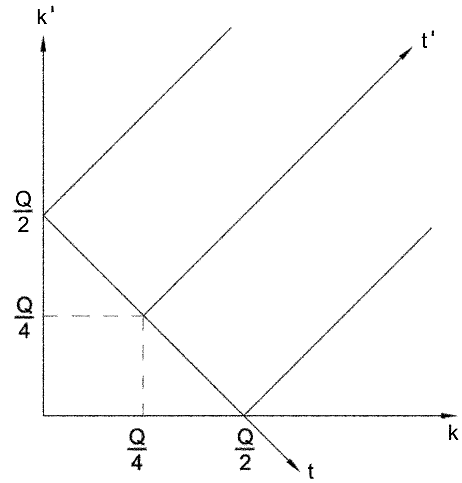

Figure 1: The integration domain, location of the point of maximal value, and transition to the new variables for the nonrelativistic case

At this transformation function $S(k,k')$ gets dependence on large parameter Q^2:

$$S^*(Q^2,t,t') = -\frac{r_0^2}{2}\left(t^2 + t'^{\,2} + \frac{Q}{\sqrt{2}}\,t' + \frac{Q^2}{8}\right) \tag{23}$$

Functions $S(Q^2,t,t')$, $g_{0C}^{00}(k\,,Q^2\,,k')$, and the boundary of the integration domain satisfy the conditions of theorem 1°, 3°, 4°. The location of the point, that satisfies the conditions (5), (6), can be obtained by a simple analysis of the function (23): $(t^0,t'^{\,0}) = (0,0)$. Let us show that this point satisfies the conditions 2° of the theorem.

It is obvious, that in the point of maximal value

$$\frac{\partial S}{\partial n} = \left.\frac{\partial S^*}{\partial t'}\right|_{(t,t')=(0,0)} = -\frac{r_0^2\,Q}{2\sqrt{2}} \neq 0\,.$$

So the condition (5) is satisfied.

Let us calculate now the B matrix from the condition (6). In our case the tangent to the domain of integration boundary vector in the point of maximal value is $\vec{\xi} = (1/\sqrt{2},-1/\sqrt{2})$, i.e. the B matrix is a number:

$$\frac{\partial^2 S}{\partial \xi^2} = \left.\frac{\partial^2 S^*}{\partial t^2}\right|_{(t,t')=(0,0)} = -r_0^2 < 0\,. \tag{24}$$

We note, that B is negative-definite, i.e. the point $(t^0,t^{0\,\prime}) = (0,0)$ is really the point of maximal value. So the point (0,0) satisfies the condition 2° of the theorem.

So integral (19) satisfies the requirements of the theorem proven in the Sec. 2. Therefore we can apply the asymptotic formula (9).

Calculating by analogy the other terms of the sum (14) we obtain asymptotic expansions of deuteron form factors in the nonrelativistic impulse approximation:

$$G_i^{NR}(Q^2) \sim e^{-\frac{r_0^2 Q^2}{16}} \sum_{m=0}^{\infty} \frac{h_{2m}^{NR}}{(2m)!}\, \Gamma(m+\frac{1}{2}), \tag{25}$$

$$h_{2m}^{NR} = \sum_{k=0}^{\infty} \frac{1}{r_0^{2m+2k+3}} \sum_{p=0}^{k} b_{kp} \left(\frac{2\sqrt{2}}{Q}\right)^{p+k+1} \left.\frac{\partial^{2m}}{\partial t^{2m}} f_i^{NR\,(k-p)}(t,Q^2,0)\right|_{t=0}, \tag{26}$$

$$b_{k0}=1,\; b_{kp}=b_{k-1\,p}-(k+p-1)b_{k-1\,p-1},\; b_{kk}=(-1)^k(2k-1)!!\,,$$

$$f_i^{NR}(t,Q^2,t') = A_i \sum_{l,l'=0,2} k^{l+2}\, k'^{\,l'+2}\, \tilde{u}_l\,(k)\;\; \tilde{g}_{0i}^{ll'}(t,Q^2,t')\; \tilde{u}_{l'}\,(k')\;\;,$$

with $k = k(t,Q^2,t')\,, k' = k'(t,Q^2,t')$, variables $t,\; t'$ are denoted in (22), $i = C,Q,M$, $A_C = 1,\; A_Q = 2M_d^2/Q^2\,,\; A_M = -M_d$,

$$f_i^{NR\,(m)}(t,Q^2,t') = \frac{\partial^m}{\partial t'^{\,m}} f_i^{NR}(t,Q^2,t')\;.$$

$\tilde{u}_{l,l'}$ is defined by equalities:

$$u_0(k) = \tilde{u}_0(k)\; e^{-\frac{r_0^2 k^2}{2}}\;,\; u_2(k) = \tilde{u}_2(k)\; k^2\; e^{-\frac{r_0^2 k^2}{2}}\;. \tag{27}$$

Let us perform now the calculation of the relativistic asymptotics of deuteron form factors. To estimate asymptotically integrals (17) we proceed analogously to nonrelativistic case, i.e. we use relativistic analogs of corresponding nonrelativistic formulas (19)-(24). Now the free relativistic charge form factor in (17) at $l = l' = 0$ is given in Ref. [8]:

$$g_{0C}^{00}(s,Q^2,s') = R(s,Q^2,s')\,Q^2\left[\,(s+s'+Q^2)\left(G_E^p(Q^2)+G_E^n(Q^2)\right)g_{CE}^{00} + \right.$$

$$\left. + \frac{1}{M}\xi(s,Q^2,s')\left(G_M^p(Q^2)+G_M^n(Q^2)\right)g_{CM}^{00}\right]\;, \tag{28}$$

$G_{E,M}^{p,n}(Q^2)$ are electric and magnetic form factors of proton and neutron respectively,

$$g_{CE}^{00} = \left(\frac{1}{2}\cos\omega_1\cos\omega_2 + \frac{1}{6}\sin\omega_1\sin\omega_2\right)\;,\quad g_{CM}^{00} = \left(\frac{1}{2}\cos\omega_1\sin\omega_2 - \frac{1}{6}\sin\omega_1\cos\omega_2\right)\;,$$

$$R(s,Q^2,s') = \frac{(s+s'+Q^2)}{\sqrt{(s-4M^2)(s'-4M^2)}}\,\frac{\vartheta(s,Q^2,s')}{[\lambda(s,-Q^2,s')]^{3/2}}\,\frac{1}{\sqrt{1+Q^2/4M^2}}\;,$$

$$\xi(s,Q^2,s') = \sqrt{ss'Q^2 - M^2\lambda(s,-Q^2,s')}\;,$$

ω_1 and ω_2 are angles of the Wigner spin rotation,

$$\omega_1 = \arctan\frac{\xi(s,Q^2,s')}{M\left[(\sqrt{s}+\sqrt{s'})^2+Q^2\right]+\sqrt{ss'}(\sqrt{s}+\sqrt{s'})}\;,$$

$$\omega_2 = \arctan\frac{\alpha(s,s')\xi(s,Q^2,s')}{M(s+s'+Q^2)\alpha(s,s')+\sqrt{ss'}(4M^2+Q^2)}\;, \tag{29}$$

where $\alpha(s,s') = 2M + \sqrt{s} + \sqrt{s'}$, $\vartheta(s,Q^2,s') = \theta(s'-s_1) - \theta(s'-s_2)$, θ is a step function, $\lambda(a,b,c) = a^2+b^2+c^2-2(ab+ac+bc)$,

$$s_{1,2} = 2M^2 + \frac{1}{2M^2}(2M^2+Q^2)(s-2M^2) \mp \frac{1}{2M^2}\sqrt{Q^2(Q^2+4M^2)s(s-4M^2)}\ .$$

To obtain relativistic asymptotic expansion we also perform transition to the new basis (shift and rotation). The function S and the boundary of the integration domain differ from nonrelativistic ones, so it is necessary to perform a special analysis. In other words, instead of change of variables (22) we perform the following replacement:

$$s = \frac{1}{\sqrt{2}}\left(t' + \frac{t}{Q}\right) + 2M^2 + M\sqrt{Q^2+4M^2}\ ,\quad s' = \frac{1}{\sqrt{2}}\left(t' - \frac{t}{Q}\right) + 2M^2 + M\sqrt{Q^2+4M^2}\ . \tag{30}$$

Then we obtain asymptotic expansion of the relativistic deuteron form factors by analogy with nonrelativistic case:

$$G_i^R(Q^2) \sim e^{-\frac{r_0^2}{4}(M\sqrt{Q^2+4M^2}-2M^2)} \sum_{m=0}^{\infty} \frac{h_{2m}^R}{(2m)!}\Gamma(m+\frac{1}{2}), \tag{31}$$

$$h_{2m}^R = \sum_{k=0}^{\infty}\sum_{p=0}^{2m} \frac{(-1)^p}{Q^{p+m-\frac{1}{2}}}\ (2p-1)!!\ C_{2p}^{2m}\ M^{m-p+\frac{1}{2}}\ \frac{2^{\frac{5}{2}k+4m-2p+\frac{9}{2}}}{r_0^{2k+2m-2p+3}}\ \frac{\partial^{2m-2p}}{\partial t^{2m-2p}} f_i^{R\,(k)}(t,Q^2,\varphi(t))\bigg|_{t=0}\ , \tag{32}$$

$$f_i^R(t,Q^2,t') = A_i \sum_{l,l'=0,2} \tilde{u}_l(k)\ g_{0i}^{ll'}(t,Q^2,t')\ \tilde{u}_{l'}(k') \frac{(s/4-M^2)^{\frac{l+1}{2}}\,(s'/4-M^2)^{\frac{l'+1}{2}}}{\sqrt[4]{s\,s'}},$$

$$f_i^{R\,(m)}(t,Q^2,t') = \frac{\partial^m}{\partial t'^{\,m}} f_i^R(t,Q^2,t')\ .$$

Functions $k = k(s)\,, k' = k'(s')$ are specified in Eq. (18), $s = s(t,Q^2,t')\,, s' = s'(t,Q^2,t')$, variables $t,\ t'$ are denoted in (30), C_{2p}^{2m} are the binomial coefficients.

Asymptotic expansions (25) and (31) are convergent power series in inverse degrees of the parameter Q with known coefficients. The asymptotic expansion of this type is obtained in this work for the first time.

One can see from formulas (25) and (31), that relativistic corrections change the behavior of form factors at high momentum transfer. In particular, exponential multiplier index is Q^2 in the nonrelativistic case, but in the relativistic case it is Q at $Q^2 \to \infty$. It seems to be a general feature of our relativistic approach to the description of composite systems, in particular, we have obtained the similar result in consideration of asymptotic behavior of the pion form factor in the composite quark model [16].

5 Asymptotics of the form factors for the conventional wave functions representation

In this Section we represent the obtained asymptotical expansions (25) and (31) in terms of initial wave functions in the left side of Eqs. (15), (18). For this representation it

is necessary to replace functions $\tilde{u}_l(k)$ by functions $u_l(k)$ in (25) and (31) using (18), (27). Keeping the main term on $1/Q$ in asymptotic expansions (25) and (31) one can obtain the next asymptotic formulas in terms of functions $u_l(k)$ and $\varphi_l(s)$ from (15), (18):

$$G_i^{NR}(Q^2) = -A_i \frac{4\sqrt{\pi}}{r_0^3 Q} \sum_{l,l'=0,2} k^2\, k'^{\,2} u_l(k)\, \tilde{g}_{0i}^{ll'}(t,Q^2,t')\, u_{l'}(k') \Bigg|_{\substack{t=0\\ t'=0}} , \tag{33}$$

$$G_i^{R}(Q^2) = -A_i \frac{8\,\sqrt{\sqrt{2}\pi M}}{r_0^3\sqrt{Q}} \sum_{l,l'=0,2} \frac{\varphi_l(s)\; g_{0i}^{ll'}(t,Q^2,t')\; \varphi_{l'}(s')}{\sqrt[4]{s\,s'}} \Bigg|_{\substack{t=0\\ t'=0}} , \quad i = C, Q, M\ . \tag{34}$$

Let us note, that similar asymptotic representation can be obtained for any finite number of terms in asymptotic expansions (25), (31).

In the modern calculations the deuteron wave functions are usually represented as a discrete superposition of Yukawa-type terms (see, e.g., [9]):

$$u_0(k) = \sqrt{\frac{2}{\pi}} \sum_j \frac{C_j}{(k^2 + m_j^2)}, \qquad u_2(k) = \sqrt{\frac{2}{\pi}} \sum_j \frac{D_j}{(k^2 + m_j^2)}, \tag{35}$$

or in the coordinate representation:

$$u_0(r) = \sum_j C_j \exp\left(-m_j\, r\right) ,$$

$$u_2(r) = \sum_j D_j \exp\left(-m_j\, r\right) \left[1 + \frac{3}{m_j\, r} + \frac{3}{(m_j\, r)^2}\right] , \tag{36}$$

$$m_j = \alpha + m_0\,(j-1) , \quad \alpha = \sqrt{M\ |\varepsilon_d|}\ .$$

Coefficients C_j, D_j, maximal value of the index j and m_0 are determined by the best fit of corresponding solution of Schrödinger equation.

The deuteron wave function analytical form (36) results in the right behavior of the wave functions at large distances:

$$u_0(r)\ \sim\ \exp(-\,\alpha\, r) , \quad u_2(r)\ \sim\ \exp(-\,\alpha\, r) \left(1 + \frac{3}{(\alpha\, r)} + \frac{3}{(\alpha\, r)^2}\right) . \tag{37}$$

The deuteron wave functions behavior at small distances:

$$u_0(r)\ \sim\ r\ , \quad u_2(r)\ \sim\ r^3\ , \tag{38}$$

is provided by imposing the following conditions on coefficients C_j and D_j:

$$\sum_j C_j = 0\ , \quad \sum_j D_j = \sum_j D_j m_j^2 = \sum_j \frac{D_j}{m_j^2} = 0\ . \tag{39}$$

Let us substitute the wave functions (35) to (33) and (34), and then obtain the first asymptotic terms of the nonrelativistic deuteron form factors:

$$G_C^{NR} \sim \frac{1}{Q^8}\frac{2^{16}}{\sqrt{\pi}r_0^3}\left[\sum_j C_j m_j^2\right]^2 \left(G_E^p(Q^2)+G_E^n(Q^2)\right)\ , \tag{40}$$

$$G_Q^{NR} \sim 3\, M_d^2 \frac{1}{Q^{12}}\frac{2^{\frac{43}{2}}}{\sqrt{\pi}r_0^3}\left[\sum_j C_j m_j^2\right]\left[\sum_j D_j m_j^4\right]\left(G_E^p(Q^2)+G_E^n(Q^2)\right)\ , \tag{41}$$

$$G_M^{NR} \sim \frac{1}{Q^8}\frac{2^{16}M_d}{\sqrt{\pi}r_0^3 M}\left[\sum_j C_j m_j^2\right]^2 \left(G_M^p(Q^2)+G_M^n(Q^2)\right)\ . \tag{42}$$

The dimensional parameter r_0 can be found from the expression for the deuteron matter radius in our deuteron model:

$$r_m^2 = \frac{1}{4}\int_0^\infty (u_0^2(r)+u_2^2(r))r^2 dr\ . \tag{43}$$

One can substitute wave functions of the form (16) into this expression. So formula (43) specifies an algebraic equation for r_0. Solution of this equation can be found numerically.

It should be pointed out that main terms of expansion of charge and magnetic form factors in (40) - (42) are determined by S-state of deuteron only. The D-wave function gives the contribution to the main term of the quadrupole from factor. Its faster decrease at $Q^2 \to \infty$ in comparison to the other form factors is a consequence of a faster decrease of a D-wave function at small distances in comparison to S-wave (38). From the mathematical point of view the type of leading terms in (40)-(42) is a consequence of conditions on the coefficients (39). The modification of these conditions obviously results in change of the main terms in (40)-(42). From these formulas it is also noticed that asymptotic expansions for the deuteron form factors contain dependence on the asymptotics of nucleon form factors.

We emphasize, that in the other deuteron asymptotics investigations only the power dependence on the transferred momentum was calculated as a rule. In the present paper we give a rigorous calculation of a multiplicative preasymptotical constant.

One can calculate relativistic asymptotics of form factors by analogy with nonrelativistic case. For this calculation we use the formulas (18), (34), (39). As a result we obtain:

$$G_{C,M}^R(Q^2)\ \sim\ \frac{Q^3}{2^{\frac{7}{2}}M^3}\,G_{C,M}^{NR}(Q^2)\ , \tag{44}$$

$$G_Q^R(Q^2)\ \sim\ \frac{Q^4}{2^{\frac{11}{2}}M^4}\,G_Q^{NR}(Q^2)\ . \tag{45}$$

Notice that asymptotic expansions (40)-(42) and (44),(45) are obtained for the first time in our work. It is interesting to compare obtained asymptotic estimations (40) - (42), (44),(45) with observable behavior of the deuteron characteristics. At present time there exists the experimental information about function $A(Q^2)$ entered the differential cross section of the elastic ed-scattering. This function is expressed in terms of the deuteron form factors [1]. The values of function $A(Q^2)$ are known up to $Q^2\ \simeq\ 6$ (GeV/c)^2. For the comparison with experimental data one needs to specify asymptotics of the nucleon form factors. It is naturally to choose for nucleon form factors the asymptotic which is predicted by the quark model [1] $G_M^{p,n}\ \sim\ 1/Q^4$. Under these conditions the power

dependence on Q^2 of the function $A(Q^2)$ coincides with experimentally observed one. The physical consequences will be examined in detail in the other paper.

6 Conclusion

The theorem defining asymptotics of multiple integrals of some special type is proved. With help of the proven theorem the asymptotic expansion of the deuteron electromagnetic form factors at $Q^2 \to \infty$ is calculated for the first time. The expansion is represented as a convergent series on in inverse powers of momentum transfer. The asymptotic of the form factors is found in terms of the conventional representation of the deuteron wave function as a discrete superposition of Yukawa-type terms. The asymptotic behavior of the form factors is calculated in the nonrelativistic impulse approximation and in the relativistic invariant impulse approximation proposed by the authors in the instant form of the Poincare-invariant quantum mechanics previously. It is established that relativistic corrections change the power dependence of the form factors on the momentum transfer at $Q^2 \to \infty$ and slow down its decrease. It is also found that relativistic effects result in the agreement of the theoretical asymptotics and the experimentally observed behavior of the structure function $A(Q^2)$ at highest achieved momentum transfers.

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